JOURNAL OF APPROXIMATION THEORY 48, 251–261 (1986)

On Localized Potential Spaces

A. CARBERY*

Department of Mathematics, University of Chicago, Chicago, Illinois 60637, U.S.A.

G. GASPER†

Department of Mathematics, Northwestern University, Evanston, Illinois 60201, U.S.A.

AND

W. TREBELS

Fb Mathematik, TH Darmstadt, D-6100 Darmstadt, West Germany

Communicated by Antoni Zygmund

Received September 13, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

Localized potential spaces are quite useful in the theory of Fourier multipliers or multipliers with respect to orthogonal expansions (see, e.g., [8, 3, 2)]: They seem to be the right setting (i) to prove end-point multiplier criteria on L^p-spaces (p near 1 or 2) and (ii) to interpolate (by the complex method) between these end-point results. Localized Bessel potential and Besov spaces have already been introduced by Strichartz [11] and by Connett and Schwartz [3], who also derived their basic properties. In this paper we want to give a concise derivation of the properties of the localized Riemann–Liouville spaces $\mathrm{RL}(q, \alpha)$, which are a slight variant of the localized potential spaces considered earlier. They turn out to coincide with the spaces $\mathrm{WBV}_{q,\alpha}$, $\alpha q > 1$, of functions of weak bounded variation considered in [6] and to coincide with the localized Riesz potential spaces $R(q, \alpha)$, $1 < q < \infty$, used in [2]; thus giving a proof of Theorem 2 in [2].

*Supported in part by the National Science Foundation.
†Supported in part by NSF Grant DMS-8403566.

0021-9045/86 $3.00

1. Definition and Basic Embedding Properties of $RL(q, \alpha)$

Fix a nonnegative bump function $\phi \in C^\infty(\mathbb{R})$ with support in the interval $[1, 2]$ and define for $\alpha > 0$, $1 \leqslant q \leqslant \infty$, the localized Riemann–Liouville spaces

$$\mathrm{RL}(q, \alpha) = \{m \in L^1_{\mathrm{loc}}(0, \infty) : \|m\|_{\mathrm{RL}(q,\alpha)} < \infty\}$$

where, using the notation $m_t(s) = m(ts)$,

$$\|m\|_{\mathrm{RL}(q,\alpha)} = \sup_{t>0} \|(\phi m_t)^{(\alpha)}\|_q \tag{1}$$

with the standard L^q-norm on $\mathbb{R}$ and where the (fractional) derivative of order α is defined for integrable functions h of compact support in $(0, \infty)$ by

$$(h^{(\alpha)})^\wedge(\sigma) = (-i\sigma)^\alpha h^\wedge(\sigma), \qquad \sigma \in \mathbb{R}, \tag{2}$$

in the distributional sense. Then we have, if $h, h^{(\alpha)} \in L^q$, that

$$h(s) = C \int_s^\infty (t - s)^{\alpha-1} h^{(\alpha)}(t)\, dt \qquad \text{a.e.} \tag{3}$$

and further, if we set $\Delta_t h(s) = h(s+t) - h(s)$, $\Delta_t^k = \Delta_t \Delta_t^{k-1}$,

$$h^{(\alpha)}(s) = C \int_0^\infty t^{-1-\alpha} \Delta_t^k\, h(s)\, dt \qquad \text{a.e., } k > \alpha, \tag{4}$$

where C is a constant independent of h.

These formulas are proved for $0 < \alpha < 1$, $k = 1$, e.g., in [6, Sect. 3], and can readily be extended to arbitrary $\alpha > 0$. In particular it follows immediately from (4) that if $h(s) = 0$ for $s \geqslant a$, then $h^{(\alpha)}(s) = 0$ for $s \geqslant a$, and that in this case the integration over $(0, \infty)$ can be replaced by that over $(0, a)$. We start the discussion of the $\mathrm{RL}(q, \alpha)$-spaces by showing that the definition is independent of a particular bump function.

Lemma 1. *If $1 \leqslant q \leqslant \infty$, $\alpha > 0$, and $\psi \in C^\infty(\mathbb{R})$ is an arbitrary nonnegative bump function with compact support in $(0, \infty)$, then*

$$\sup_{t>0} \|(\psi m_t)^{(\alpha)}\|_q \tag{5}$$

is equivalent to the norm (1) on $\mathrm{RL}(q, \alpha)$.

Proof. First assume $0 < \alpha < 1$ and observe that by (3)

$$\|\phi m_t\|_q \leqslant \int_0^2 r^{\alpha-1} \| (\phi(\cdot + r)\, m_t(\cdot + r))^{(\alpha)}\|_q\, dr$$
$$\leqslant C\|m\|_{\mathrm{RL}(q,\alpha)}. \tag{6}$$

Since ψ has compact support there exist finitely many $t_i > 0$ such that

$$\tilde{\phi}(s) = \sum_{i=1}^{N} \phi(t_i s) \geqslant \varepsilon > 0$$

on supp ψ and $\tilde{\phi} \geqslant \psi$; hence $\psi/\tilde{\phi} \in C_0^\infty(\mathbb{R}_+)$. Then, by (4),

$$\|(\psi m_t)^{(\alpha)}\|_q \leqslant C \left\| \int_0^\infty r^{-1-\alpha} \Delta_r [(\psi/\tilde{\phi})\, \tilde{\phi} m_t]\, dr \right\|_q$$
$$\leqslant C \sup_{r>0} \|\tilde{\phi}(\cdot + r)\, m_t(\cdot + r)\|_q$$
$$+ C\|\psi/\tilde{\phi}\|_\infty \left\| \int_0^\infty r^{-1-\alpha} \Delta_r \tilde{\phi} m_t\, dr \right\|_q$$
$$\leqslant C\|m\|_{\mathrm{RL}(q,\alpha)}$$

uniformly in $t > 0$, i.e., (5) is dominated by (1). The converse inequality is proved analogously and the extension to all $\alpha > 0$ follows the same pattern; one has to additionally use part (i) of the following theorem which describes the Sobolev-type embedding behavior of the $\mathrm{RL}(q, \alpha)$ spaces.

THEOREM 1. *Let $0 \leqslant \beta < \alpha$. Then, in the sense of continuous embedding,*

(i) $\mathrm{RL}(q, \alpha) \subset \mathrm{RL}(q, \beta)$, $1 \leqslant q \leqslant \infty$;

(ii) $\mathrm{RL}(q, \alpha) \subset \mathrm{RL}(r, \beta)$, $\alpha - \beta = 1/q - 1/r$, $1 < q < r < \infty$;

(iii) $\mathrm{RL}(q, \alpha) \subset L^\infty$, $\alpha > 1/q$, $1 \leqslant q \leqslant \infty$;

(iv) $\mathrm{RL}(1, \alpha) \subset \mathrm{RL}(q, \beta)$, $1 - 1/q < \alpha - \beta$, $1 \leqslant q < \infty$.

Proof. (i) By (2), $h^{(\beta)} = \mathscr{F}^{-1}[(-i\sigma)^{\beta-\alpha}] * h^{(\alpha)}$ in the distribution sense which in the present situation implies (cf. also [6, Sect. 3]) that

$$\|(\phi m_t)^{(\beta)}\|_q = C \left\| \int_0^2 r^{\alpha-\beta-1} (\phi(r + \cdot)\, m_t(r + \cdot))^{(\alpha)} dr \right\|_q$$
$$\leqslant C\|m\|_{\mathrm{RL}(q,\alpha)} \int_0^2 r^{\alpha-\beta-1} dr.$$

(ii) This is an immediate consequence of the Hardy–Littlewood theorem on fractional integration (see [7, p. 288]), which says that $(-i\sigma)^{\beta-\alpha}$ is the symbol of a bounded operator from L^q into L^r.

(iii) By (3) and Hölder's inequality we obtain

$$|\phi(s)\, m_t(s)| \leqslant C\|m\|_{\mathrm{RL}(q,\alpha)} \left(\int_0^2 r^{(\alpha-1)q'} dr \right)^{1/q'},$$

where the last constant is finite if $\alpha > 1/q$. Since ϕ is a nonnegative bump function we thus have

$$\|m\|_\infty \leqslant C \sup_{t>0} \|\phi m_t\|_\infty \leqslant C\|m\|_{\mathrm{RL}(q,\alpha)}.$$

(iv) By a slight variant of a theorem of Bernstein (cf. [9, Theorem 5]) one can show that $(-i\sigma)^\beta(z+(-i\sigma)^\alpha)^{-1}$ is the Fourier transform of an L^q-function if $\alpha-\beta > 1/q'$, $1 < q' \leqslant \infty$ (here the constant $z \in C$ is chosen in such a way that $z+(-i\sigma)^\alpha$ never vanishes for fixed α).

Hence, the convolution theorem and Minkowski's inequality give

$$\begin{aligned}&\|(\phi m_t)^{(\beta)}\|_q \\ &\quad \leqslant \left\| \mathscr{F}^{-1}\left[\frac{(-i\sigma)^\beta}{z+(-i\sigma)^\alpha}\right]\right\|_q \{\|z\phi m_t\|_1 + \|m\|_{\mathrm{RL}(1,\alpha)}\}\end{aligned}$$

from which the assertion follows by (6).

2. Connections with Other Localized Potential Spaces

Again let h be an integrable function with compact support in $(0, \infty)$. Define the Riesz derivative of order α by

$$[D_{\mathrm{R}}^\alpha h]^\wedge(\sigma) = |\sigma|^\alpha h^\wedge(\sigma), \qquad \sigma \in \mathbb{R},$$

in the distributional sense and, analogously, the Bessel derivative by

$$[D_{\mathrm{B}}^\alpha h]^\wedge(\sigma) = (1+\sigma^2)^{\alpha/2} h^\wedge(\sigma).$$

Lemma 2. *If* $1 < q < \infty$ *and* $\alpha > 0$, *then*

$$\sup_{t>0} \|D_{\mathrm{R}}^\alpha[\phi m_t]\|_q \tag{7}$$

and

$$\sup_{t>0} \|D_{\mathrm{B}}^{\alpha}[\phi m_t]\|_q \tag{8}$$

are equivalent to the norm (1) *on* RL(q, α).

Proof. Since

$$(-i\sigma)^{\alpha} = |\sigma|^{\alpha} \left(\cos \frac{\pi}{2}\alpha - i(\operatorname{sgn}\sigma) \sin \frac{\pi}{2}\alpha\right)$$

and the Hilbert transform (with symbol $-i \operatorname{sgn}\sigma$) is a bounded operator on L^q, $1 < q < \infty$, the equivalence of (1) with (7) is obvious.

Lemma 2 in [10, p. 133] shows that

$$\|D_{\mathrm{B}}^{\alpha}[\phi m_t]\|_q \approx \|\phi m_t\|_q + \|D_{\mathrm{R}}^{\alpha}[\phi m_t]\|_q,$$

i.e., (8) majorizes (7). Conversely, by (6), the term $\|\phi m_t\|_q$ can be dominated by the norm (1) which, by the above, is dominated by (7), completing the proof.

In [6], two of the authors introduced the space $\mathrm{WBV}_{q,\alpha}$ of functions of weak bounded variation which turns out to coincide with RL(q, α), $\alpha q > 1$, as we will see in Theorem 2. Let us point out that multiplier theorems are most naturally proved in the RL(q, α)-context, whereas the $\mathrm{WBV}_{q,\alpha}$-conditions are easier to verify.

For a locally integrable function h and $0 < \delta < 1$ we define [6] the fractional integral

$$\begin{aligned} I_{\omega}^{\delta}(h)(t) &= \frac{1}{\Gamma(\delta)} \int_t^{\omega} (s-t)^{\delta-1} h(s)\, ds, && 0 < t < \omega \\ &= 0, && t \geq \omega \end{aligned}$$

and the fractional derivatives

$$\begin{aligned} h^{(\delta)}(t) &= \lim_{\omega \to \infty} -\frac{d}{dt} I_{\omega}^{1-\delta}(h)(t), \\ h^{(\alpha)}(t) &= (-d/dt)^{[\alpha]} h^{(\alpha-[\alpha])}(t), \qquad \alpha > 0 \end{aligned} \tag{9}$$

whenever the right sides exist ($[\alpha]$ denotes the integer part of α). We assume from now on that $I_{\omega}^{1-\delta}(h) \in AC_{\text{loc}}$, $\omega > 0$, if $\delta = \alpha - [\alpha] > 0$ and $h^{(\delta)}, \ldots, h^{(\alpha-1)} \in AC_{\text{loc}}$ if $\alpha \geq 1$ when we speak of (fractional) derivatives in the sense of (9).

Then the definitions (2) and (9) coincide for integrable functions h with compact support in $(0, \infty)$ (see [6, Sect. 3]) and therefore we utilize the same notation for a fractional derivative. Now the space of functions of weak bounded variation is described as follows:

$$\mathrm{WBV}_{q,\alpha} = \{m \in L^\infty \cap C(0, \infty): \|m\|_{q,\alpha} < \infty, \alpha > 0, 1 \leqslant q \leqslant \infty\}$$

where

$$\begin{aligned} \|m\|_{q,\alpha} &= \|m\|_\infty + \sup_{m \in \mathbb{Z}} \left(\int_{2^{m-1}}^{2^m} |t^\alpha m^{(\alpha)}(t)|^q \frac{dt}{t} \right)^{1/q}, \qquad q < \infty \\ &= \|m\|_\infty + \operatorname*{ess\,sup}_{t>0} |t^\alpha m^{(\alpha)}(t)|, \qquad q = \infty. \end{aligned}$$

THEOREM 2. *If $\alpha > 1/q$ and $1 \leqslant q \leqslant \infty$, then*

$$\mathrm{RL}(q, \alpha) = \mathrm{WBV}_{q,\alpha}$$

with equivalent norms.

Proof. If $m \in \mathrm{WBV}_{q,\alpha}$ then

$$\begin{aligned} \int |(\phi m_t)^{(\alpha)}(s)|^q ds &= C t^{\alpha q - 1} \int |(\phi_{1/t} m)^{(\alpha)}(s)|^q ds \\ &= C t^{\alpha q - 1} \left(\int_{-\infty}^{t/4} + \int_{t/4}^{4t} \right) |(\phi_{1/t} m)^{(\alpha)}(s)|^q ds \\ &\equiv I_1 + I_2, \qquad \text{say.} \end{aligned}$$

A slight modification of the proof of Lemma 10 in [6] shows that $m \in \mathrm{WBV}_{q,\alpha}$ if and only if $\phi_{1/t} m \in \mathrm{WBV}_{q,\alpha}$ for each $t > 0$ and $\sup_{t>0} \|\phi_{1/t} m\|_{q,\alpha} < \infty$. Moreover,

$$\|m\|_{q,\alpha} \approx \sup_{t>0} \|\phi_{1/t} m\|_{q,\alpha}. \tag{10}$$

Then

$$\begin{aligned} I_2 &\leqslant C \int_{t/4}^{4t} |s^\alpha (\phi_{1/t} m)^{(\alpha)}(s)|^q \frac{ds}{s} \\ &\leqslant C \|\phi_{1/t} m\|_{q,\alpha}^q \leqslant C \|m\|_{q,\alpha}^q. \end{aligned}$$

Concerning I_1, observe that

$$(\phi_{1/t}m)^{(\alpha)}(s) = \int_{t/2}^{4t} (r-s)^{-\alpha-1}\phi_{1/t}(r)\, m(r)\, dr$$
$$\leqslant C\|\phi_{1/t}m\|_\infty \left(\frac{t}{2}-s\right)^{-\alpha}, \qquad s \leqslant \frac{t}{4},$$

by definition (9), and therefore

$$I_1 \leqslant Ct^{\alpha q-1}\|m\|_\infty^q \int_{-\infty}^{t/4} \left(\frac{t}{2}-s\right)^{-\alpha q} ds \leqslant C\|m\|_\infty^q.$$

Thus $\mathrm{WBV}_{q,\alpha} \subset \mathrm{RL}(q, \alpha)$.

Now let $m \in \mathrm{RL}(q, \alpha)$. In view of (10) we have to consider the integrals

$$I_{R,t} = \int_R^{2R} |s^\alpha(\phi_{1/t}m)^{(\alpha)}(s)|^q \frac{ds}{s}, \qquad R, t > 0.$$

Clearly $I_{R,t} = 0$ when $R > 4t$. If $R < t/8$, then as above

$$I_{R,t} \leqslant CR^{\alpha q-1}\|\phi_{1/t}m\|_\infty^q \int_R^{2R} \left(\frac{t}{2}-s\right)^{-\alpha q} ds$$
$$\leqslant C\|\phi m_t\|_\infty^q \leqslant C\|m\|_{\mathrm{RL}(q,\alpha)}^q$$

by Theorem 1(iii). Finally, if $t/8 \leqslant R \leqslant 4t$,

$$I_{R,t} \leqslant Ct^{\alpha q-1} \int_R^{2R} |(\phi_{1/t}m)^{(\alpha)}(s)|^q ds \leqslant C\|m\|_{\mathrm{RL(q,\alpha)}}^q.$$

Since the required AC_{loc} smoothness conditions for $\phi_{1/t}m$ to be in $\mathrm{WBV}_{q,\alpha}$ are easily verified (cf. [6, Sect. 3]), this completes the proof.

3. Interpolation Properties of $\mathrm{RL}(q, \alpha)$

We reduce the problem to the analogous result for Bessel potential spaces

$$L_\alpha^q = \{f \in L^q(\mathbb{R}) \colon \|f\|_{L_\alpha^q} := \|\mathscr{F}^{-1}[(1+\xi^2)^{\alpha/2} f^\wedge]\|_q < \infty\}.$$

Using Calderón's [1] lower and upper complex interpolation methods, it is well known that

$$L_\alpha^q = [L_{\alpha_0}^{q_0}, L_{\alpha_1}^{q_1}]_\theta = [L_{\alpha_0}^{q_0}, L_{\alpha_1}^{q_1}]^\theta, \qquad \alpha_0 \neq \alpha_1,\ 1 < q_0, q_1 < \infty, \qquad (11)$$

where

$$\left(\frac{1}{q}, \alpha\right) = (1-\theta)\left(\frac{1}{q_0}, \alpha_0\right) + \theta\left(\frac{1}{q_1}, \alpha_1\right), \qquad 0 < \theta < 1. \tag{12}$$

Analogously, we have

THEOREM 3. *If* $\alpha_1 > \alpha_0 > 0$, $1 < q_0, q_1 < \infty$, *and* (12) *holds, then*

$$\mathrm{RL}(q, \alpha) = [\mathrm{RL}(q_0, \alpha_0), \mathrm{RL}(q_1, \alpha_1)]^\theta.$$

In order to prove Theorem 3, we first need some preliminary results. Let $L^\infty(L^q_\alpha)$ denote the Banach space of L^q_α-valued measurable functions $F(t)$, $t > 0$, such that

$$\|F\|_{L^\infty(L^q_\alpha)} = \operatorname*{ess\,sup}_{t>0} \|F(t)\|_{L^q_\alpha} < \infty.$$

From [1, Sect. 13.6] it follows that

$$L^\infty(L^q_\alpha) = [L^\infty(L^{q_0}_{\alpha_0}), L^\infty(L^{q_1}_{\alpha_1})]^\theta, \tag{13}$$

provided θ and the α's and q's satisfy the conditions in Theorem 3,

With $[f]_\phi$ denoting the equivalence class of functions g such that $\phi f = \phi g$ a.e., define the Banach spaces of equivalence classes $[f]_\phi$ of functions f with $\phi f \in L^q_\alpha$ by

$$L^q_{\alpha,\phi} = \{[f]_\phi : \|[f]_\phi\|_{L^q_{\alpha,\phi}} = \|\phi f\|_{L^q_\alpha} < \infty\}.$$

Let $\pi: L^q_\alpha \to L^q_{\alpha,\phi}$ be defined by $\pi(f) = [f]_\phi$. Then π is continuous and it induces a continuous map

$$\pi^*: L^\infty(L^q_\alpha) \to L^\infty(L^q_{\alpha,\phi}), \qquad \pi^*(F) = [F]_\phi, \tag{14}$$

where $[F]_\phi(t) = \pi(F(t)) = [F(t)]_\phi$ and

$$\|[F]_\phi\|_{L^\infty(L^q_{\alpha,\phi})} = \operatorname*{ess\,sup}_{t>0} \|[F]_\phi(t)\|_{L^q_{\alpha,\phi}}.$$

LEMMA 3. *If* $\alpha > 0$ *and* $1 < q < \infty$, *then there exist continuous maps* $\mathscr{A}$ *and* $\mathscr{B}$ *such that*

$$\mathscr{A}: \mathrm{RL}(q, \alpha) \to L^\infty(L^q_\alpha), \tag{15}$$

$$\mathscr{B}: L^\infty(L^q_{\alpha,\phi}) \to \mathrm{RL}(q, \alpha), \tag{16}$$

$$\mathscr{B} \circ \pi^* \circ \mathscr{A} = I_{\mathrm{RL}(q,\alpha)} \qquad (\textit{identity map on } \mathrm{RL}(q, \alpha)), \tag{17}$$

and

$$\|\mathscr{A}f\|_{L^\infty(L^q_\alpha)} \approx \|f\|_{\mathrm{RL}(q,\alpha)}, \qquad f \in \mathrm{RL}(q, \alpha).$$

Proof. First, for $f \in \mathrm{RL}(q, \alpha)$, define

$$(\mathscr{A}f(t))(x) = \phi(x)\, f(xt).$$

The fact that $\mathscr{A}f(t)$ is L^q_α-valued measurable follows from the observation that, since L^q_α is reflexive [12, p. 198], it suffices by Pettis' theorem [5, Theorem 11, p. 149] to show that $\|\mathscr{A}f(t) - \mathscr{A}f(s)\|_{L^q_\alpha} \to 0$ as $s \to t$, which is easily verified. It follows from Lemma 2 that

$$\|\mathscr{A}f\|_{L^\infty(L^q_\alpha)} = \operatorname*{ess\,sup}_{t>0} \|\phi(\cdot)\, f(\cdot\, t)\|_{L^q_\alpha} \approx \|f\|_{\mathrm{RL}(q,\alpha)}.$$

Next, for $[F]_\phi \in L^\infty(L^q_{\alpha,\phi})$ define

$$\mathscr{B}([F]_\phi)(x) = c \int_0^\infty \phi\left(\frac{x}{t}\right)(F(t))\left(\frac{x}{t}\right)\frac{dt}{t},$$

where $c = (\int_0^\infty \phi^2(u)\, du/u)^{-1}$. Notice that if $[F]_\phi = [G]_\phi$, then $\phi(x)(F(t))(x) = \phi(x)(G(t))(x)$ and so $\mathscr{B}$ is well-defined. Since, for $f \in \mathrm{RL}(q, \alpha)$,

$$\begin{aligned}(\mathscr{B} \circ \pi^* \circ \mathscr{A})\, f(x) &= \mathscr{B}([\phi(\cdot)\, f(\cdot\, t)]_\phi)(x) \\ &= c \int_0^\infty \phi\left(\frac{x}{t}\right) \phi\left(\frac{x}{t}\right) f(x) \frac{dt}{t} \\ &= c f(x) \int_0^\infty \phi^2(u) \frac{du}{u} = f(x),\end{aligned}$$

it is clear that $\mathscr{B} \circ \pi^* \circ \mathscr{A} = I_{\mathrm{RL}(q,\alpha)}$. To see that $\mathscr{B}$ is continuous, first observe that

$$\begin{aligned}\|\mathscr{B}([F]_\phi)\|_{\mathrm{RL}(q,\alpha)} &\leqslant C \sup_{u>0} \left\| \int_0^\infty \phi\left(\frac{\cdot\, u}{t}\right)(F(t))\left(\frac{\cdot\, u}{t}\right)\frac{dt}{t}\,\phi(\cdot)\right\|_{L^q_\alpha} \\ &= C \sup_{u>0} \left\| \int_0^\infty D^\alpha_{\mathrm{B}} \left\{\phi\left(\frac{\cdot\, u}{t}\right)(F(t))\left(\frac{\cdot\, u}{t}\right)\phi(\cdot)\right\} \frac{dt}{t}\right\|_q.\end{aligned}$$

For fixed u, $\phi(xu/t)(F(t))(xu/t)\phi(x) \equiv 0$ unless $u/2 \leqslant t \leqslant 2u$. Hence, by the integral Minkowski inequality,

$$\|\mathscr{B}([F]_\phi)\|_{\mathrm{RL}(q,\alpha)}$$

$$\leqslant C \sup_{u>0} \operatorname*{ess\,sup}_{u/2\leqslant t\leqslant 2u} \left\| D_{\mathrm{B}}^{\alpha}\left\{\phi\left(\frac{\cdot\, u}{t}\right)(F(t))\left(\frac{\cdot\, u}{t}\right)\phi(\cdot)\right\}\right\|_q \int_{u/2}^{2u}\frac{dt}{t}$$

$$= C \sup_{u>0} \operatorname*{ess\,sup}_{u/2\leqslant t\leqslant 2u} \left\| \phi\left(\frac{\cdot\, u}{t}\right)(F(t))\left(\frac{\cdot\, u}{t}\right)\phi(\cdot)\right\|_{L_\alpha^q}$$

$$\leqslant C \sup_{1/2\leqslant s\leqslant 2} \operatorname*{ess\,sup}_{t>0} \|\phi(\cdot)(F(t))(\cdot)\,\phi(\cdot s)\|_{L_\alpha^q}$$

$$\leqslant C \operatorname*{ess\,sup}_{t>0} \|\phi F(t)\|_{L_\alpha^q} = C\|[F]_\phi\|_{L^\infty(L^q_{\alpha,\phi})}.$$

This completes the proof of Lemma 3 and leads us to our

Proof of Theorem 3. Suppose that the hypotheses in Theorem 3 are satisfied. The fact that

$$\mathrm{RL}(q,\alpha)\subseteq[\mathrm{RL}(q_0,\alpha_0),\mathrm{RL}(q_1,\alpha_1)]^\theta$$

with a continuous inclusion follows from Lemma 3 by using the mappings

$$\mathrm{RL}(q,\alpha)\xrightarrow{\mathscr{A}} L^\infty(L_\alpha^q)=[L^\infty(L_{\alpha_0}^{q_0}),L^\infty(L_{\alpha_1}^{q_1})]^\theta$$
$$\xrightarrow{\pi^*}[L^\infty(L_{\alpha_0,\phi}^{q_0}),L^\infty(L_{\alpha_1,\phi}^{q_1})]^\theta$$
$$\xrightarrow{\mathscr{B}}[\mathrm{RL}(q_0,\alpha_0),\mathrm{RL}(q_1,\alpha_1)]^\theta$$

where $\mathscr{A}$, π^*, and $\mathscr{B}$ are the maps induced by (15), (14), and (16). On the other hand, from

$$[\mathrm{RL}(q_0,\alpha_0),\mathrm{RL}(q_1,\alpha_1)]^\theta\xrightarrow{\mathscr{A}}[L^\infty(L_{\alpha_0}^{q_0}),L^\infty(L_{\alpha_1}^{q_1})]^\theta=L^\infty(L_\alpha^q)$$
$$\xrightarrow{\pi^*}L^\infty(L_{\alpha,\phi}^q)\xrightarrow{\mathscr{B}}\mathrm{RL}(q,\alpha)$$

we find that

$$[\mathrm{RL}(q_0,\alpha_0),\mathrm{RL}(q_1,\alpha_1)]^\theta\subseteq\mathrm{RL}(q,\alpha)$$

with a continuous inclusion, which completes the proof.

It should be noted that since, by [6, Theorem 3], $\mathrm{WBV}_{q,\alpha}$ is equivalent to the localized Bessel potential space $S(q,\alpha)$ when $\alpha>1/q$, $1<q<\infty$, and since Connett and Schwartz [4] have shown that the spaces $[S(q_0,\alpha_0), S(q_1,\alpha_1)]_\theta$ and $S(q,\alpha)$ are not equivalent unless $q_0=q_1$ and $\alpha_0=\alpha_1$, the space $[\mathrm{RL}(q_0,\alpha_0),\mathrm{RL}(q_1,\alpha_1)]^\theta$ in Theorem 3 cannot be replaced by the space $[\mathrm{RL}(q_0,\alpha_0),\mathrm{RL}(q_1,\alpha_1)]_\theta$.

During the preparation of this paper we received a preprint by Muckenhoupt, Wheeden, and Young entitled "Weighted L^p Multipliers," which also considers the $\mathrm{RL}(q, \alpha)$ spaces.

REFERENCES

1. A. P. CALDERÓN, Intermediate spaces and interpolation, the complex method, *Studia Math.* **24** (1964), 113–190.
2. A. CARBERY, G. GASPER, AND W, TREBELS, Radial Fourier multipliers of $L^p(\mathbb{R}^2)$, *Proc. Nat. Acad. Sci. U.S.A.* **81** (1984), 3254–3255.
3. W. C. CONNETT AND A. L. SCHWARTZ, The theory of ultraspherical multipliers, *Mem. Amer. Math. Soc.* **183** (1977).
4. W. C. CONNETT AND A. L. SCHWARTZ, A failure of stability under complex interpolation, *Canad. J. Math.* **29** (1977), 1167–1170.
5. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators," Part I Interscience, New York, 1958.
6. G. GASPER AND W. TREBELS, A characterization of localized Bessel potential spaces and applications to Jacobi and Hankel multipliers, *Studia Math.* **65** (1979), 243–278.
7. G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, "Inequalities," Cambridge Univ. Press, Cambridge, 1964.
8. L. HÖRMANDER, Estimates for translation invariant operators in L^p-spaces, *Acta Math.* **104** (1960), 93–140.
9. W. R. MADYCH, On Littlewood–Paley functions, *Studia Math.* **50** (1974), 43–63.
10. E. M. STEIN, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, N.J., 1970.
11. R. S. STRICHARTZ, Multipliers for spherical harmonic expansions, *Trans. Amer. Math. Soc.* **167** (1972), 115–124.
12. H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators," North-Holland, Amsterdam, 1978.